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Suitability of Capital Allocations for Performance Measurement

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Abstract

Capital allocation principles are used in various contexts in which a risk capital or a cost of an aggregate position has to be allocated among its constituent parts. We study capital allocation principles in a performance measurement framework. We introduce the notation of suitability of allocations for performance measurement and show under different assumptions on the involved risk measures that there exist suitable allocation methods. The existence of certain suitable allocation principles generally is given under rather strict assumptions on the underlying risk measure. Therefore we show, with a reformulated definition of suitability and in a slightly modified setting, that there is a known suitable allocation principle that does not require any properties of the underlying risk measure. Additionally we extend a previous characterization result from the literature from a mean-risk to a reward-risk setting. Formulations of this theory are also possible in a game theoretic setting.

Keywords Risk capital allocation, subgradient allocation, cost allocation, suitability for performance measurement

1 Introduction

Capital allocation principles are used in various contexts in which a risk capital or a cost of an aggregate position has to be allocated among its constituent parts. The aggregate position, for instance, can be a portfolio of credit derivatives, the profit, cost or turnover of a firm or simply the outcome of a cooperative game. The importance of capital allocation principles in credit portfolio modeling is outlined in Kalkbrener et al. (2004). Capital allocations can even be used to construct tax schemes for pollution emission, see Cheridito and Kromer (2011). In any case one is interested in how to allocate (distribute) the aggregate cost/profit among the units which are involved in generating this cost/profit.

The main financial mathematics literature dealing with this topic follows a structural approach to capital allocations and studies the properties of specific allocation methods. Kalkbrener

(2005), for instance, studies in detail the properties of the gradient allocation and places them in context with the properties of the underlying coherent risk measure. Other contributions, like Denault (2001), Fischer (2003) and McNeil et al. (2005), similarly focus on the gradient allocation and study its properties. This allocation method only exists if the underlying risk measure is Gâteaux-differentiable. This requirement is a strong one and several risk measures in general fail to be Gâteaux-differentiable. For an overview of this aspect we refer to Cheridito and Kromer (2011). For the existence of the subgradient allocation that was introduced in Delbaen (2000), we need weaker requirements. Continuity of the underlying risk measure, to name one, is a sufficient condition to ensure the existence of a subgradient allocation. There are also allocation principles in the literature that do not require any properties of the risk measure and still have some desirable structural properties. A recent example for such allocation principles are the ordered contribution allocations from Cheridito and Kromer (2011) or the with-without allocation from Merton and Perold (1993) and Matten (1996).

In this work we will focus on another aspect of capital allocation methods. These can be used as important tools in performance measurement for portfolios and firms and consequently for portfolio optimization. The literature that studies capital allocations in this framework, to our knowledge, narrows down to the works of Tasche (2004) and Tasche (2008). He studies the gradient allocation principle based on Gâteaux-differentiable risk measures and uses mean-risk ratios as performance measures. He introduces a definition of suitability for performance measurement for capital allocations and characterizes the gradient allocation as the unique allocation that is suitable for performance measurement. This is a different approach to this topic compared to the structural approaches we previously mentioned. We will take up and extend this approach in different directions. We will show in Theorem 3.2, with a slightly modified definition of suitability, that we can relax the requirement of Gâteaux-differentiability of the underlying risk measure to a (local) continuity property. This indeed preserves the existence of capital allocations that are suitable for performance measurement, but we loose the uniqueness. As we will show, any subgradient allocation is suitable for performance measurement. If the underlying risk measure is Gâteaux-differentiable, the subgradient allocation is unique and reduces to the gradient allocation. In this case our result reduces to the result of Tasche (2004). In another setting, where we restrict our performance measure to portfolios that do not allow arbitrage opportunities and irrationality we are able to work with a stronger formulation of suitability for performance measurement and to derive in this setting an existence and uniqueness result in Theorem 3.9. In the following we will work with classes of reward-risk ratios, which were recently introduced in Cheridito and Kromer (2012). These are generalizations of mean-risk ratios from Tasche (2004). We will show that a further modification of the definition of suitability for performance measurement allows us to carry over the whole approach to a cooperative game theory setting and to consider cost allocation methods. This setting has the benefit that we do not require any properties of the underlying cost function to ensure existence of suitable cost allocation methods. Moreover, there is a direct connection to the risk-capital based setting that allows us to apply these results directly to risk measures without requiring the risk measure to have any specific properties.

The outline of the paper is as follows. In Section 2 we introduce our notation and give a short overview of the capital allocation methods that will be relevant for us in the remaining work. We will work with reward-risk ratios as performance measures. These will be introduced in Section 3. In this Section we furthermore introduce the definition of suitability for performance measurement with reward-risk ratios and present our main results. In Section 4 we modify the setting from Section 3 to a cooperative game theory setting and show in this new framework

the existence of capital allocations that are suitable for performance measurement.

2 Notation, definitions and important properties of risk measures and capital allocations

A financial position, the value of a firm or the outcome of a cooperative game will be modeled as a real valued random variable X from L^p, $p \in [1,\infty]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As usual, we identify two random variables if they agree $\mathbb{P}$-a.s. Inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense.

We will work with convex and with coherent risk measures $\rho : L^p \to (-\infty, \infty]$. Coherent risk measures were introduced and studied in Artzner et al. (1999). We will call a functional from L^p to $(-\infty, \infty]$ a coherent risk measure if it has the following four properties.

(M) *Monotonicity:* $\rho(X) \leq \rho(Y)$ for all $X, Y \in L^p$ such that $X \geq Y$.

(T) *Translation property:* $\rho(X + m) = \rho(X) - m$ for all $X \in L^p$ and $m \in \mathbb{R}$.

(S) *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in L^p$.

(P) *Positive homogeneity:* $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in L^p$ and $\lambda \in \mathbb{R}_+$.

From the properties (S) and (P) it follows that coherent risk measures are convex, accordingly they satisfy the property

(C) *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in L^p$ and $\lambda \in (0,1)$.

Functionals with the properties (M), (T) and (C) are called convex risk measures. These were introduced and studied in Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). We will denote the domain of a convex risk measure ρ with $\operatorname{dom}\rho = \{X \in L^p \mid \rho(X) < \infty\}$. ρ is a *proper* convex risk measure if the domain of ρ is nonempty. Since it only makes sense to allocate a *finite* risk capital of an aggregate position among its constituent parts we will only work with aggregate positions X from the domain of ρ. We know from the extended Namioka-Klee Theorem from Frittelli and Biagini (2010) that proper convex risk measures are continuous on the interior of their domain. From this it follows that finite valued convex risk measures on L^p are continuous on L^p, $1 \leq p \leq \infty$. In particular proper convex risk measures on L^∞ are automatically finite and continuous on L^∞. In the following we will need the concept of the subdifferential of a convex risk measure on L^p. The subdifferential of a convex risk measure at $X \in \operatorname{dom}\rho$, $1 \leq p < \infty$ is the set

$$\partial\rho(X) = \{\xi \in L^q \mid \rho(X+Y) \geq \rho(X) + \mathbb{E}[\xi Y] \quad \forall Y \in L^p\}, \tag{2.1}$$

where q is such that $1/q + 1/p = 1$ and L^q is the dual space of L^p. We will call ρ subdifferentiable at X if $\partial\rho(X)$ is nonempty. We know from Proposition 3.1 in Ruszczynski and Shapiro (2006) that ρ with the properties (M) and (C) is continuous and subdifferentiable on the interior of its domain. For $p = \infty$ we can consider the weak*-subdifferential introduced in Delbaen (2000)

$$\partial\rho(X) = \left\{\xi \in L^1 \;\middle|\; \rho(X+Y) \geq \rho(X) + \mathbb{E}[\xi Y] \quad \forall Y \in L^\infty\right\}$$

In contrast to (2.1) this set can be empty at some points in L^∞ even if the risk measure is finite valued and thus continuous on L^∞. For more details about the weak*-subdifferential we

refer to Delbaen (2000). It is known that $\partial\rho(X)$ is a singleton if and only if ρ is Gâteaux-differentiable, see for instance Zalinescu (2002). Then $\mathbb{E}(\xi Y)$ for $\xi \in \partial\rho(X)$ and $Y \in L^p$ is the Gâteaux-derivative at X in the direction of Y, ie,

$$\mathbb{E}(\xi Y) = \lim_{h\to 0} \frac{\rho(X+hY)-\rho(X)}{h}. \tag{2.2}$$

Furthermore we know that if ρ is Gâteaux-differentiable at X then ρ is continuous in X.

Consider now the aggregate position $X = \sum_{i=1}^n X_i \in \operatorname{dom}\rho,\ n \in \mathbb{N}$. It is of interest for several reasons outlined in the introduction, to allocate the aggregate risk capital $k = \rho(X)$ to the X_i, $i = 1, \dots, n$, which are involved in generating this risk capital. We will denote a capital allocation as a vector of real numbers $k = (k_1, \dots, k_n) \in \mathbb{R}^n$. The simplest capital allocation $k \in \mathbb{R}^n$ is the individual allocation given by

$$k_i = \rho(X_i),\ i = 1, \dots, n.$$

If the underlying risk measure is subadditive, which is property (S) of coherent risk measures, then the individual allocation overestimates the total risk, ie,

$$\rho\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \rho(X_i) = \sum_{i=1}^n k_i.$$

A reasonable allocation principle should satisfy the above inequality as equality. This is then usually called full allocation property or efficiency of the allocation,

$$\rho\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n k_i. \tag{2.3}$$

Since the individual allocation only depends on the risk of the individual asset or subportfolio X_i and does not take into account the risk of the whole portfolio one is certainly interested in other allocation principles. Another candidate, that additionally takes into account the risk of the aggregate position is the with-without allocation,

$$k_i = \rho(X) - \rho(X - X_i). \tag{2.4}$$

This allocation principle, known from Merton and Perold (1993) and Matten (1996) will be important in Section 4. For risk measures with property (S) both principles are related through

$$\rho(X) - \rho(X - X_i) \le \rho(X_i),$$

such that the individual allocation is an upper bound for the with-without allocation. Nevertheless, both allocation principles in general do not satisfy the full allocation property if they are based on coherent risk measures.

If the underlying risk measure is convex and the subdifferential of ρ at X is nonempty, then for any $\xi \in \partial\rho(X)$ we can consider the subgradient allocation,

$$k_i = \mathbb{E}(\xi X_i). \tag{2.5}$$

This allocation principle was introduced in Delbaen (2000) with the weak*-subdifferential for coherent risk measures on L^∞. If $\partial\rho(X)$ is a singleton the subgradient allocation reduces to

the gradient allocation which is also known as the Euler allocation, see Tasche (2008). Then $k_i = \mathbb{E}(\xi X_i)$ is the Gâteaux-derivative at X in the direction of X_i, ie,

$$k_i = \lim_{h_i \to 0} \frac{\rho(X + h_i X_i) - \rho(X)}{h_i}. \tag{2.6}$$

It is known from Euler's Theorem on positively homogeneous functions (see Tasche (2008), Theorem A.1) that it satisfies the full allocation property if the risk measure is Gâteaux-differentiable and positively homogeneous.

Measuring the risk of a unit without taking into account the risk of the whole entity (which corresponds to the individual allocation), besides the lack of the full allocation property, has also conceptual drawbacks in different frameworks. Thus we suggest to use other candidates of capital allocations just as for performance measurement as for risk measurement. As a consequence we will study in the following Sections which capital allocations incorporate the risk of the whole entity and are suitable for performance measurement.

Before we move to the next section we will report in short the results of Tasche (2004) which form the basis for this suitability-for-performance-measurement approach and which we will generalize in the following. Let $X = \sum_{i=1}^n X_i$ be a portfolio consisting of n subportfolios X_i, $i = 1, \dots, n$. Let $\emptyset \neq U \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be an open set and define $\rho_X : U \to \mathbb{R}$, $\rho_X(u) := \rho(\sum_{i=1}^n u_i X_i)$. With $m_i = \mathbb{E}[X_i]$ we denote similarly $\mathbb{E}_X(u) = \mathbb{E}[\sum_{i=1}^n u_i X_i] = \sum_{i=1}^n u_i m_i$. Here each $u \in U$ represents a portfolio. Then a capital allocation is identified in this setting by a map

$$k = (k_1, \dots, k_n) : U \to \mathbb{R}^n.$$

Tasche (2004) considers in his approach mean-risk ratios, which are also called RORAC (return on adjusted risk capital) ratios. These are defined by $RORAC_X : U \to \mathbb{R}$,

$$RORAC_X(u) := \frac{\mathbb{E}_X(u)}{\rho_X(u)}, \tag{2.7}$$

where in Tasche (2004) irrational portfolios ($u \in U$ such that $\mathbb{E}_X(u) \leq 0$ and $\rho_X(u) > 0$) and arbitrage portfolios ($u \in U$ such that $\mathbb{E}_X(u) > 0$ and $\rho_X(u) \leq 0$) are completely excluded from the approach. In this setting with mean-risk ratios as measures of performance Tasche (2004) provides the following definition of suitability of capital allocations for performance measurement.

Definition 2.1 (compare to Definition 3 in Tasche (2004))**.** *A vector field $k : U \to \mathbb{R}^n$ is called suitable for performance measurement with ρ_X if it satisfies the following conditions:*

(i) For all $m \in \mathbb{R}^n$, $u \in U$ and $i \in \{1, \dots, n\}$ the inequality

$$m_i \rho_X(u) > k_i(u) \mathbb{E}_X(u) \tag{2.8}$$

implies that there is an $\varepsilon > 0$ such that for all $h \in (0, \varepsilon)$ we have

$$RORAC_X(u + he_i) > RORAC_X(u) > RORAC_X(u - he_i). \tag{2.9}$$

(ii) For all $m \in \mathbb{R}^n$, $u \in U$ and $i \in \{1, \dots, n\}$ the inequality

$$m_i \rho_X(u) < k_i(u) \mathbb{E}_X(u) \tag{2.10}$$

implies that there is an $\varepsilon > 0$ such that for all $h \in (0, \varepsilon)$ we have

$$RORAC_X(u + he_i) < RORAC_X(u) < RORAC_X(u - he_i). \tag{2.11}$$

Note the strong requirement that inequalities (2.8) and (2.10) have to be true for all $m \in \mathbb{R}^n$.

With this definition of suitability for performance measurement Tasche (2004) provides the following existence and uniqueness result.

Theorem 2.2 (compare to Theorem 1 in Tasche (2004))**.** *Let $\rho_X : U \to \mathbb{R}$ be a function that is partially differentiable in U with continuous derivatives. Let $k = (k_1, \dots, k_n) : U \to \mathbb{R}^n$ be a continuous vector field. Then the vector field k is suitable for performance measurement with ρ_X if and only if*

$$k_i(u) = \frac{\partial}{\partial u_i} \rho_X(u), \quad i = 1, \dots, n, \, u \in U. \tag{2.12}$$

In contrast to Tasche (2004) we will consider reward-risk ratios that were recently introduced in Cheridito and Kromer (2012) as measures of performance. This leads to the next section.

3 Suitability for performance measurement with RRRs

In this Section we will examine the connection of reward-risk ratios to capital allocations that are suitable for performance measurement. Let us first introduce reward-risk ratios.

As performance measures in this work we consider reward-risk ratios (RRRs in short) $\alpha : L^p \to \mathbb{R} \cup \{+\infty\}$ of the form

$$\alpha(X) = \begin{cases} 0, & \text{if } \theta(X) \leq 0 \text{ and } \rho(X) > 0, \\ +\infty, & \text{if } \theta(X) > 0 \text{ and } \rho(X) \leq 0, \\ \frac{\theta(X)}{\rho(X)}, & \text{else.} \end{cases} \tag{3.13}$$

This means that we will measure the performance of the aggregate position X by a ratio with a risk measure $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ in the denominator and a reward measure $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$ in the numerator. Hereby $0/0$ and ∞/∞ are understood to be 0. A reward measure $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$ in our setting is a functional with property (P) and property

(S2) *Superadditivity:* $\theta(X + Y) \geq \theta(X) + \theta(Y)$ for all $X, Y \in L^p$,

Classes of reward measures with these properties were recently introduced in Cheridito and Kromer (2012) as robust reward measures and distorted reward measures. We will shortly repeat their definition here. Let $\mathcal{P}$ be a non-empty set of probability measures that are absolutely continuous with respect to $\mathbb{P}$. The robust reward measure is defined by

$$\theta(X) = \inf_{Q \in \mathcal{P}} \mathbb{E}_Q[X]. \tag{3.14}$$

θ defined in (3.14) obviously satisfies (S2) and (P). Through the set $\mathcal{P}$ we can introduce ambiguity such that the elements of the set $\mathcal{P}$ could describe the beliefs of different traders. This allows a much more general approach to the performance measurement framework than simply choosing the expectation for the nominator of the reward-risk ratio. A subclass of (3.14) are distorted reward measures. Let $\varphi : [0, 1] \to [0, 1]$ be a non-decreasing function satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. It induces the distorted probability $\mathbb{P}_\varphi[A] := \varphi \circ \mathbb{P}[A]$. If φ is convex then the Choquet integral defined by

$$\mathbb{E}_\varphi[X] := \int_0^\infty \mathbb{P}_\varphi[X > t] dt + \int_{-\infty}^0 (\mathbb{P}_\varphi[X > t] - 1) dt, \tag{3.15}$$

is superadditive and positively homogeneous. Thus it satisfies (S2) and (P). With different choices for φ we can model different attitudes of agents towards rewards. For additional information about RRRs induced by coherent or convex risk measures in conjunction with these reward measures we refer to Cheridito and Kromer (2012). Most important for us in this work are the properties (S2) and (P) and the existence of interesting classes of reward measures with these properties. These classes obviously contain the expectation. Thus all statements in the following Sections are also true for mean-risk ratios (see e.g. (2.7)) as performance measures.

Now we will examine the connection of reward-risk-ratios to capital allocations that are suitable for performance measurement. The following definition will clarify what we mean with suitability for performance measurement with reward-risk ratios of type (3.13).

Definition 3.1. *A capital allocation $k \in \mathbb{R}^n$ is suitable for performance measurement in $X \in \operatorname{dom} \rho$ with a reward-risk-ratio $\alpha : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \leq p \leq \infty$, if for any decomposition of $X = \sum_{i=1}^n X_i$ it satisfies the following conditions:*

1. *If $\theta(X) \geq 0$ then for any $i \in \{1, \ldots, n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$*

$$\frac{\theta(X_i)}{k_i} \geq \frac{\theta(X)}{\rho(X)} \tag{3.16}$$

   implies

$$\alpha(X) \geq \alpha(X - h_i X_i) \tag{3.17}$$

   and

$$\frac{\theta(X_i)}{k_i} \leq \frac{\theta(X)}{\rho(X)} \tag{3.18}$$

   implies

$$\alpha(X) \geq \alpha(X + h_i X_i) \tag{3.19}$$

2. *If $\theta(X) \leq 0$ then for any $i \in \{1, \ldots, n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$* (3.16) *implies*

$$\alpha(X + h_i X_i) \geq \alpha(X) \tag{3.20}$$

   and (3.18) *implies*

$$\alpha(X - h_i X_i) \geq \alpha(X) \tag{3.21}$$

The motivation behind this definition of suitability for performance measurement is similar to the one in Tasche (2004) and Tasche (2008). Evidently, (3.16) and (3.18) relate the performance of the aggregate position X, measured by $\alpha(X)$, to the performance of each unit X_i, $i = 1, \ldots, n$, which is involved in generating the risk capital $\rho(X)$. The performance of each unit X_i, $i = 1, \ldots, n$, is measured by $\theta(X_i)/k_i$ (assume for the sake of simplicity that $k_i > 0$ and $\rho(X) > 0$) and only those capital allocations $k = (k_1, \ldots, k_n)$ are suitable for performance measurement that send the right signals to the agent. The right signals in this context are represented by (3.17) and (3.19) (in connection to (3.16) and (3.18)) and imply not to reduce the capital of position X_i if the standalone performance of X_i is better than the performance of the aggregate position X. On the other hand a worse standalone performance of X_i should send the signal not to invest additional capital in this position. Summarized, this means

- $\alpha(X) \geq \alpha(X - h_i X_i)$ not to reduce the capital in the position X_i
- $\alpha(X) \leq \alpha(X - h_i X_i)$ not to increase the capital in the position X_i

- $\alpha(X + h_iX_i) \geq \alpha(X)$ increase the capital in the position X_i
- $\alpha(X - h_iX_i) \geq \alpha(X)$ reduce the capital in the position X_i

We will see in Subsection 3.1 that there is a stronger formulation of suitability for performance measurement, but it also requires much stronger additional properties of the underlying risk measure to guarantee the existence of suitable capital allocations.

For further statements we will need the following technical assumption on the underlying risk measure.

(A) For an aggregate position $X \in \operatorname{dom}\rho$ and any decomposition $\sum_{i=1}^n X_i = X$ of it there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$ we have

- if $\rho(X) > 0$ then $\rho(X - h_iX_i) > 0$ and $\rho(X + h_iX_i) > 0$,
- if $\rho(X) < 0$ then $\rho(X - h_iX_i) < 0$ and $\rho(X + h_iX_i) < 0$.

for any $i \in \{1, \dots, n\}$.

It should be noted that convex risk measures that are continuous in $X \in \operatorname{dom}\rho$, automatically satisfy assumption (A) in X.

Now we can examine which properties a capital allocation should have to be suitable for performance measurement with a reward-risk ratio α in $X \in \operatorname{dom}\rho$.

Theorem 3.2. *Let $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \leq p \leq \infty$, be a reward measure with the properties (S2) and (P). Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \leq p \leq \infty$, be a map that satisfies condition (A) in $X \in \operatorname{dom}\rho$ and let α be a reward risk ratio of type* (3.13)*. Then any capital allocation $k \in \mathbb{R}^n$ that satisfies*

$$h_ik_i \leq \rho(X + h_iX_i) - \rho(X) \quad \textit{and} \tag{3.22}$$

$$h_ik_i \geq \rho(X) - \rho(X - h_iX_i) \quad \textit{for all} \quad h_i \in (0, \varepsilon_i), \tag{3.23}$$

for any decomposition of $X = \sum_{i=1}^n X_i$ is suitable for performance measurement with α in X.

Proof. Consider first the case where numerator and denominator have the same signs. Let $\theta(X) \geq 0$. With assumption (A) for ρ in $X \in \operatorname{dom}\rho$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$ we have $\rho(X)\rho(X + h_iX_i) > 0$ and $\rho(X)\rho(X - h_iX_i) > 0$ for any $i \in \{1, \dots, n\}$. From (S2) and (P) we get $\theta(X - h_iX_i) \leq \theta(X) - h_i\theta(X_i)$. This, together with (3.16) and (3.23) leads to

$$\begin{aligned} &\theta(X - h_iX_i)\rho(X) - \theta(X)\rho(X - h_iX_i) \\ \leq &\theta(X)(\rho(X) - h_ik_i - \rho(X - h_iX_i)) \leq 0, \end{aligned}$$

which leads to $\alpha(X) \geq \alpha(X - h_iX_i)$. By analogous steps we get from (3.18) and (3.22) the property $\alpha(X) \geq \alpha(X + h_iX_i)$.

Now let $\theta(X) \leq 0$. Then again with assumption (A) for ρ in $X \in \operatorname{dom}\rho$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$ we have $\rho(X)\rho(X + h_iX_i) > 0$ and $\rho(X)\rho(X - h_iX_i) > 0$ for any $i \in \{1, \dots, n\}$. From (S2), (P), (3.16) and (3.22) we get

$$\begin{aligned} &\theta(X + h_iX_i)\rho(X) - \theta(X)\rho(X + h_iX_i) \\ \geq &\theta(X)(\rho(X) + h_ik_i - \rho(X + h_iX_i)) \geq 0, \end{aligned}$$

which leads to $\alpha(X) \le \alpha(X + h_i X_i)$. By analogous steps we get from (3.18) and (3.23) the property $\alpha(X) \le \alpha(X - h_i X_i)$.

Let us now consider the case with $\theta(X) \le 0$ and $\rho(X) > 0$. Here we have $\alpha(X) = 0$ and from $\theta(X_i)\rho(X) \ge \theta(X)k_i$ it follows that

$$\alpha(X + h_i X_i) \ge \alpha(X)\frac{\rho(X) + h_i k_i}{\rho(X + h_i X_i)} = 0.$$

From $\theta(X_i)\rho(X) \le \theta(X)k_i$ it follows that

$$\alpha(X - h_i X_i) \ge \alpha(X)\frac{\rho(X) - h_i k_i}{\rho(X - h_i X_i)} = 0.$$

The remaining relevant case, in which $\theta(X) > 0$ and $\rho(X) < 0$, leads to $\alpha(X) = +\infty$ and since $\alpha(X + h_i X_i) \le +\infty$ and $\alpha(X - h_i X_i) \le +\infty$ we have finished the proof. □

In Theorem 3.2, except for condition (A), we didn't require any further properties like coherency or convexity from the risk measure ρ. We have only used the superadditivity and the positive homogeneity of θ, which are the properties (S2) and (P). The result nevertheless does not guarantee the existence of allocations that satisfy the required properties of this theorem. However it gives us a suitability-criterion, which is easy to verify and thus gives us the opportunity to check whether a capital allocation is suitable for performance measurement or not, without requiring the risk measure to have any properties except (A). For instance we can immediately see that the with-without allocation does not satisfy both of the properties (3.22) and (3.23). For the following corollary that tells us which capital allocations satisfy (3.22) and (3.23) and thus are suitable for performance measurement we will need the nonemptiness of the subdifferential of ρ at X.

Corollary 3.3. *Let $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a reward measure with the properties (S2) and (P). Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a convex risk measure with nonempty subdifferential at $X \in \operatorname{dom}\rho$. Let α be a reward-risk ratio of type* (3.13) *and consider any decomposition of $X = \sum_{i=1}^n X_i$. If (A) is true for ρ at $X \in \operatorname{dom}\rho$ then for any $\xi \in \partial\rho(X)$ the subgradient capital allocation $k \in \mathbb{R}^n$ defined by*

$$k_i := \mathbb{E}[\xi X_i], \quad \textit{for any} \quad i \in \{1, \ldots, n\} \tag{3.24}$$

is suitable for performance measurement with α in X.

Proof. Since $\partial\rho(X)$ is nonempty there exits $\xi \in \partial\rho(X)$ for $X \in \operatorname{dom}\rho \subseteq L^p$, $1 \le p \le \infty$ such that $k_i = \mathbb{E}[\xi X_i]$ for each $i \in \{1, \ldots, n\}$ obviously satisfies (3.22) for any decomposition of X. Furthermore for any decomposition $(X_1, \ldots, X_n)$ of $X = \sum_{i=1}^n X_i$ it satisfies

$$\rho(X - h_i X_i) - \rho(X) \ge \mathbb{E}[\xi(-h_i X_i)]$$

for any $i \in \{1, \ldots, n\}$ and any $h_i \in (0, \varepsilon_i)$. This is equivalent to (3.23). Thus with Theorem 3.2 for any $\xi \in \partial\rho(X)$ any subgradient capital allocation $k_i = \mathbb{E}[\xi X_i]$, $\xi \in \partial\rho(X)$, $i = 1, \ldots, n$, is suitable for performance measurement with α at X. □

As outlined in Section 2 we know that convex risk measures on L^p, $1 \le p < \infty$ are continuous and subdifferentiable on the interior of their domain. Thus if the aggregate position X is from the interior of the domain of ρ condition (A) and the nonemptiness of the subdifferential of ρ at X are automatically satisfied. This leads to the next Corollary.

Corollary 3.4. *Let $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a reward measure with the properties (S2) and (P). Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a convex risk measure and let α be a reward-risk ratio of type* (3.13)*. If the aggregate position $X = \sum_{i=1}^n X_i$ is from the interior of the domain of ρ then the subdifferential of ρ at X is nonempty and any subgradient allocation defined in* (3.24) *is suitable for performance measurement with α in $X \in int(dom\, \rho)$.*

If we directly assume that the underlying convex risk measure is continuous at the aggregate position X, then assumption (A) is satisfied, the subdifferential of ρ at X is nonempty and thus the subgradient allocation exists and is suitable for performance measurement with a reward-risk ratio α. Since the continuity of a convex risk measure ρ on L^p follows from the finiteness of ρ on L^p, $1 \le p \le \infty$ we can formulate the following Corollary.

Corollary 3.5. *Let $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a reward measure with the properties (S2) and (P). Let $\rho : L^p \to \mathbb{R}$, $1 \le p \le \infty$, be a convex risk measure and let α be a reward-risk ratio of type* (3.13)*. Then the subdifferential of ρ at X is nonempty for any $X \in L^p$ and any subgradient allocation is suitable for performance measurement with α at any $X \in L^p$.*

As already mentioned in Section 2 Gâteaux-differentiability of ρ at X leads to continuity of ρ at X and to the uniqueness of the subgradient of ρ at X. In this case the Gâteaux-derivative is the only element of the subdifferential of ρ at X at thus the capital allocation defined in (2.6) is suitable for performance measurement with a reward-risk ratio α at X. This allows us to formulate the final Corollary from Theorem 3.2.

Corollary 3.6. *Let $\theta : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a reward measure with the properties (S2) and (P). Let α be a reward-risk ratio of type* (3.13)*. If the risk measure $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux-differentiable at the aggregate position X, then the gradient allocation, defined by*

$$k_i = \lim_{h_i \to 0} \frac{\rho(X + h_i X_i) - \rho(X)}{h_i}, \tag{3.25}$$

is suitable for performance measurement with α in X.

Proof. This follows directly from Corollary 3.3 and the reasoning from Section 2. $\square$

Example 3.7. *Distortion-exponential risk measures from Tsanakas and Desli (2003) are defined as*

$$\rho_{\varphi,a}(X) = \frac{1}{a} \ln \mathbb{E}_\varphi(\exp(-aX)), \tag{3.26}$$

where $a > 0$ and $\mathbb{E}_\varphi[\cdot]$ is the Choquet integral from (3.15) *with a concave map φ. Let φ be differentiable. Then we know from Proposition 2 in Tsanakas (2009) that $\rho_{\varphi,a}$ is Gâteaux-differentiable at X, if and only if F_X^{-1} is strictly increasing. The gradient allocation in this case is given by*

$$k_i = \frac{\mathbb{E}[X_i e^{-aX} \varphi'(1 - F_X(X)]}{\mathbb{E}[e^{-aX} \varphi'(1 - F_X(X))]},$$

where F_X^{-1} is the quantile function

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(p) \ge p\}.$$

Note that with $\varphi(t) = t$ this class of risk measures covers the entropic risk measure.

If we consider distortion risk measures that are defined as Choquet-integrals with a concave map φ, see (3.15)*, then from Carlier and Dana (2003) the following result holds. Let φ be*

differentiable. Then $\mathbb{E}_\varphi$ is Gâteaux-differentiable if and only if F_X^{-1} is strictly increasing. Then the gradient allocation is given by

$$k_i = \mathbb{E}[X_i \varphi'(1 - F_X(X))].$$

Gâteaux-differentiable risk measures allow us to formulate stronger results regarding suitability for performance measurement. Thus we will deal with this aspect in a separate Subsection.

3.1 Suitability for performance measurement with Gâteaux-differentiable reward and risk measures

One way of dealing with portfolios that lead to *arbitrage opportunities* ($\theta(X) > 0$ and $\rho(X) \leq 0$) or are *irrational* ($\theta(X) \leq 0$, $\rho(X) > 0$) is to set the performance measure for such portfolios to $+\infty$ respectively 0. This is the way we followed in the previous Section. Another way is to restrict the performance measure to portfolios that do not allow for such possibilities. This is the direction we will follow in this Subsection. Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set such that for each $u \in U$ either

$$\rho_X(u) := \rho\left(\sum_{i=1}^n u_i X_i\right) > 0 \quad \text{and} \quad \theta_X(u) := \theta\left(\sum_{i=1}^n u_i X_i\right) > 0, \tag{3.27}$$

or

$$\rho_X(u) := \rho\left(\sum_{i=1}^n u_i X_i\right) < 0 \quad \text{and} \quad \theta_X(u) := \theta\left(\sum_{i=1}^n u_i X_i\right) < 0. \tag{3.28}$$

Furthermore let $U \subset \mathbb{R}^n$ be such that $\rho\left(\sum_{i=1}^n u_i X_i\right) \in \mathbb{R}$ and $\theta\left(\sum_{i=1}^n u_i X_i\right) \in \mathbb{R}$. Analogously to the previous Section we define

$$\alpha_X(u) := \frac{\theta_X(u)}{\rho_X(u)} \tag{3.29}$$

and denote a capital allocation, which is here a vector field, by

$$k = (k_1, \ldots, k_n) : U \to \mathbb{R}^n.$$

In this setting, as can be seen from the notation above, each $u \in U$ represents a portfolio. Here we fix an arbitrary decomposition of X and the only assumption that will be needed for all the following results is that θ_X and ρ_X are partially differentiable with continuous derivatives in some $u \in U$. This corresponds to Gâteaux-differentiability of θ and ρ at some aggregate position X.

Now, following the lines of Tasche (2004), we can state a stronger definition of suitability for performance measurement that fits this framework.

Definition 3.8. *Let e_j denote the j-th unit vector. We call an allocation principle k* suitable for performance measurement with α_X *in $u \in U$ if the following two conditions hold:*

1. *For any $i \in \{1, \ldots, n\}$ there is some $\varepsilon_i > 0$ such that*

$$\frac{\frac{\partial \theta_X(u)}{\partial u_i}}{k_i(u)} > \frac{\theta_X(u)}{\rho_X(u)} \tag{3.30}$$

   implies

$$\alpha_X(u + s e_i) > \alpha_X(u) > \alpha_X(u - s e_i). \tag{3.31}$$

   for all $s \in (0, \varepsilon_i)$.

2. For any $i \in \{1, \dots, n\}$ there is some $\varepsilon_i > 0$ such that

$$\frac{\frac{\partial \theta_X(u)}{\partial u_i}}{k_i(u)} < \frac{\theta_X(u)}{\rho_X(u)} \tag{3.32}$$

implies

$$\alpha_X(u - se_i) > \alpha_X(u) > \alpha_X(u + se_i). \tag{3.33}$$

for all $s \in (0, \varepsilon_i)$.

Note the difference between our definition of suitability for performance measurement with α_X and the Definition 3 of Tasche (2004). Definition 3 of Tasche (2004) requires the capital allocation k to satisfy (3.30) and (3.32) for all linear functions θ with $\theta(0) = 0$ and all portfolios $u \in U$. In our formulation we restrict this requirement to a specific reward-risk ratio α_X and a specific portfolio $u \in U$.

Now we can state and prove in a different way our version of Theorem 1 in Tasche (2004). This version fits the requirements of our definition of suitability for performance measurement with α_X in $u \in U$ in conjunction with reward-risk ratios instead of mean-risk ratios. The following result characterizes the gradient allocation in this performance measurement framework.

Theorem 3.9. *Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set such that (3.27) and (3.28) are true and let $\rho_X : U \to \mathbb{R}$ be partially differentiable in $u \in U$ with continuous derivatives. Furthermore let $\theta_X : U \to \mathbb{R}$ be partially differentiable in $u \in U$ and let $k = (k_1, \dots, k_n) : U \to \mathbb{R}^n$ be continuous in $u \in U$. Let α_X be a reward-risk ratio of type (3.29). Then we have the following two statements:*

(a) The gradient allocation, defined by

$$k_i(u) := \frac{\partial \rho_X(u)}{\partial u_i}, \quad i = 1, \dots, n,\ u \in U \tag{3.34}$$

is suitable for performance measurement with α_X in $u \in U$.

(b) If a capital allocation $k : U \to \mathbb{R}^n$ is suitable for performance measurement with α_X in $u \in U$ for any θ_X that is partially differentiable in $u \in U$, then it is the gradient allocation from (3.34).

Proof. Let us start with (a). Consider the partial derivative of α_X in u_i,

$$\frac{\partial \alpha_X(u)}{\partial u_i} = \rho_X(u)^{-2} \left(\frac{\partial \theta_X(u)}{\partial u_i} \rho_X(u) - \theta_X(u) \frac{\partial \rho_X(u)}{\partial u_i} \right) \tag{3.35}$$

With (3.34), we get

$$\frac{\partial \alpha_X(u)}{\partial u_i} = \rho_X(u)^{-2} \left(\frac{\partial \theta_X(u)}{\partial u_i} \rho_X(u) - \theta_X(u) k_i(u) \right).$$

(3.30) now leads to $\left(\frac{\partial \theta_X(u)}{\partial u_i} \rho_X(u) - \theta_X(u) k_i(u) \right) > 0$ and therefore it follows that

$$\frac{\partial \alpha_X(u)}{\partial u_i} > 0.$$

This gives us the existence of $\varepsilon_i > 0$ such that (3.31) holds for all $s \in (0, \varepsilon_i)$, $i = 1, \dots, n$. The proof of condition 2. in Definition 3.8 works analogously and (3.34) is suitable for performance measurement with α_X in $u \in U$.

Let us now prove (b). Let k be suitable for performance measurement for any θ_X that is partially differentiable in $u \in U$. We want to show, that k corresponds to (3.34). Assume without any restriction, that

$$k_i(u) < \frac{\partial \rho_X(u)}{\partial u_i}, \quad i \in \{1, \dots, n\}. \tag{3.36}$$

Since k is suitable for performance measurement with α_X in $u \in U$ for **any** $\theta_X : U \to \mathbb{R}$ that is partially differentiable in $u \in U$ we can simply choose $\theta_X^t : U \to \mathbb{R}$ to be $\theta_X^t(u) = t\rho_X(u)$ for $t > 0$. This function is by assumption partially differentiable in $u \in U$ for each $t > 0$. Assumption (3.36) provides us now with the following

$$\begin{aligned} \frac{\partial \theta_X^t(u)}{\partial u_i}\rho_X(u) - k_i(u)\theta_X^t(u) &= t\frac{\partial \rho_X(u)}{\partial u_i}\rho_X(u) - k_i(u)t\rho_X(u) \\ &= t\rho_X(u)\left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u)\right). \end{aligned}$$

Now, dependent on the sign of ρ_X, we either have

$$t\rho_X(u)\left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u)\right) > 0$$

which should lead to

$$\alpha_X(u + se_i) > \alpha_X(u) > \alpha_X(u - se_i),$$

or we have

$$t\rho_X(u)\left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u)\right) < 0,$$

which should lead to

$$\alpha_X(u + se_i) < \alpha_X(u) < \alpha_X(u - se_i),$$

since k is suitable for performance measurement with α_X in $u \in U$. But we have $\alpha_X(u) = \frac{\theta_X^t(u)}{\rho_X(u)} = \frac{t\rho_X(u)}{\rho_X(u)} = t$ and hence

$$\alpha_X(u + se_i) = \alpha_X(u) = \alpha_X(u - se_i) \quad \text{or} \quad \frac{\partial \alpha_X(u)}{\partial u_i} = 0,$$

which is a contradiction to the suitability assumption. $\square$

4 Game theoretic approach to suitability for performance measurement

In this Section we will follow a cooperative game theory approach. For details about cooperative games, see for instance Shapley (1953); Hart and Kurz (1983); Faigle and Kern (1992) and Branzei et al. (2005). Let $N = \{1, \dots, n\}$ be a finite set of players. Denote by 2^N the set of all possible subsets of N. Here, similar to Subsection 3.1 we will restrict our attention to

coalitions of players that do not allow for arbitrage gains or are irrational. Thus let $G \subset 2^N$ be such that for each $S \in G$ we either have

$$\vartheta(S) > 0 \quad \text{and} \quad c(S) > 0 \tag{4.37}$$

or

$$\vartheta(S) < 0 \quad \text{and} \quad c(S) < 0 \tag{4.38}$$

for functions $\vartheta : G \to \mathbb{R}$ and $c : G \to \mathbb{R}$ such that $\vartheta(\emptyset) = c(\emptyset) = 0$. For each coalition of players $S \in G$, the number $c(S)$ denotes the cost and $\vartheta(S)$ denotes the reward caused by the group of players in S. Furthermore let $\gamma : G \to (0, \infty)$,

$$\gamma(S) := \frac{\vartheta(S)}{c(S)}, \; S \in G,$$

denote the performance ratio between reward and cost caused by $S \in G$. In this context we will denote an allocation as a function $\kappa : G \to \mathbb{R}^n$, where $\kappa_i(S)$ specifies how much of the cost $c(S)$ is allocated to each player $i \in S$.

Now we can state a definition of suitability for performance measurement with γ for allocations κ that fits in this new cooperative game context.

Definition 4.1. *We will call a cost allocation κ* suitable for performance measurement with γ *and a coalition of players $S \in G$ if the following two conditions hold:*

1. *For any $i \in N \setminus S$*
$$\frac{\vartheta(\{i\})}{\kappa_i(S)} > \frac{\vartheta(S)}{c(S)} \tag{4.39}$$
implies that
$$\gamma(S \cup \{i\}) > \gamma(S). \tag{4.40}$$

2. *For any $i \in N \setminus S$*
$$\frac{\vartheta(\{i\})}{\kappa_i(S)} < \frac{\vartheta(S)}{c(S)} \tag{4.41}$$
implies that
$$\gamma(S \cup \{i\}) < \gamma(S). \tag{4.42}$$

This definition of suitability differs from our previous definitions. Here, from the viewpoint of the coalition of players S, we are interested in cost allocation principles that give us the right information whether to add a player i to our coalition S or not. If the standalone performance of player i (measured by $\vartheta(\{i\})/\kappa_i(S)$) is better than the performance of the whole coalition S then coalition S should add player i to improve their performance. On the other hand, if the standalone performance of player i is worse than the performance of S then adding this player to the coalition should worsen the performance of S.

Remark 4.2. *The connection to a risk-capital setting can be established by defining*

$$\vartheta(S) := \theta\left(\sum_{i\in S} X_i\right) \quad \text{and} \quad c(S) := \rho\left(\sum_{i\in S} X_i\right), \quad S \in G, \tag{4.43}$$

for a risk measure $\rho : L^p \to \mathbb{R}$ and a reward measure $\theta : L^p \to \mathbb{R}$. Then $\vartheta(\{i\})$ corresponds to $\vartheta(\{i\}) = \theta(X_i)$ and we will see later on in Corollary 4.4 that an appropriate candidate for a cost allocation $\kappa(S) = (\kappa_1(S), \dots, \kappa_n(S))$ can be specified as

$$\kappa_i(S) = \rho\Big(\sum_{j\in S} X_j + X_i\Big) - \rho\Big(\sum_{j\in S} X_j\Big). \tag{4.44}$$

What we would like to emphasize with this remark is that the results of this Section can be transferred to a risk-measure-based setting in an obvious way by (4.43) and (4.44).

In cooperative game theory reward functions like ϑ (characteristic functions of a game) are often assumed to satisfy certain properties like superadditivity, which is in this framework the property

$$\vartheta(S \cup T) \geq \vartheta(S) + \vartheta(T) \quad \text{for all} \quad S, T \subseteq N,\ S \cap T = \emptyset.$$

For the following Proposition we will assume that the reward function ϑ satisfies a stronger property, namely the additivity property

$$\vartheta(S \cup \{i\}) = \vartheta(S) + \vartheta(\{i\}) \tag{4.45}$$

for all $i \in N$ and all $S \subseteq N \setminus \{i\}$. We can see immediately that this property is equivalent to

$$\vartheta(S \cup T) = \vartheta(S) + \vartheta(T) \quad \text{for any } S, T \subseteq N,\ S \cap T = \emptyset.$$

This property enables us to state conditions for allocations to be suitable for performance measurement with γ and a coalition of players $S \in G$ in a cooperative game theory framework. But we will not need ϑ to satisfy property (4.45) for all $i \in N$ and all $S \subseteq N \setminus \{i\}$. We will only need the restriction to a specific coalition of players $S \in G$ and any $i \in N \setminus S$.

Proposition 4.3. *If the reward function ϑ satisfies the additivity property* (4.45) *for the coalition of players $S \in G$ and any $i \in N \setminus S$ then any cost allocation κ that satisfies*

$$\kappa_i(S) \geq c(S \cup \{i\}) - c(S) \quad \textit{if } c(S \cup \{i\}) > 0 \quad \textit{and} \tag{4.46}$$

$$\kappa_i(S) \leq c(S \cup \{i\}) - c(S) \quad \textit{if } c(S \cup \{i\}) < 0 \tag{4.47}$$

for any $i \in N \setminus S$ is suitable for performance measurement with γ and the coalition of players $S \in G$.

Proof. From (4.39) together with the additivity property (4.45) of ϑ we get

$$\begin{aligned}\gamma(S \cup \{i\}) - \gamma(S) &= \frac{\vartheta(S \cup \{i\})c(S) - \vartheta(S)c(S \cup \{i\})}{c(S \cup \{i\})c(S)} \\ &> \frac{\vartheta(S)[c(S) + \kappa_i(S) - c(S \cup \{i\})]}{c(S \cup \{i\})c(S)} \\ &= \gamma(S) \cdot \frac{c(S) + \kappa_i(S) - c(S \cup \{i\})}{c(S \cup \{i\})}.\end{aligned}$$

With (4.46) and (4.47) we get

$$\gamma(S \cup \{i\}) - \gamma(S) > 0.$$

Now, starting with (4.41) leads with the same arguments to

$$\gamma(S \cup \{i\}) - \gamma(S) < 0,$$

what finally proves the statement. $\square$

An obvious candidate for a capital allocation that is suitable for performance measurement with γ and the coalition of players S is the marginal contribution allocation defined by

$$\kappa_i(S) := c(S \cup \{i\}) - c(S). \tag{4.48}$$

Thus a straightforward consequence of Proposition 4.3 is the following Corollary.

Corollary 4.4. *If the reward function ϑ satisfies the additivity property* (4.45) *for the coalition of players $S \in G$ and any $i \in N \setminus S$ then the marginal contribution allocation defined in* (4.48) *is suitable for performance measurement with γ and the coalition of players $S \in G$.*

Useful properties a cost allocation in this framework can have are

Efficiency: $c(N) = \sum_{i=1}^{n} \kappa_i(N)$

Symmetry: $\kappa_i(N) = \kappa_j(N)$ if $c(\{i\} \cup S) = c(\{j\} \cup S)$ for all $S \subseteq N \setminus \{i, j\}$

Dummy Player Property: $\kappa_i(N) = c(\{i\})$ if $c(\{i\} \cup S) = c(\{i\}) + c(S)$ for every subset $S \subseteq N \setminus \{i\}$.

By restricting the whole framework of this section to the set $S = N \setminus \{i\}$ the marginal contribution allocation corresponds to the so called *with-without allocation*, introduced in Merton and Perold (1993) in a risk-capital context, see (2.4). The with-without allocation has the symmetry and the dummy player property, but in general it fails to have the efficiency property. To achieve efficiency for this allocation principle a simple normalization procedure,

$$k_i = \frac{\rho(X) - \rho(X - X_i)}{\sum_{j=1}^{n} \rho(X) - \rho(X - X_j)} \rho(X),$$

can be performed, but this is only possible if $\sum_{j=1}^{n} \rho(X) - \rho(X - X_j)$ is nonzero.

The formulation of suitability for performance measurement in this Section has a specific advantage. We have shown that there exists an allocation principle that is suitable for performance measurement with a performance measure γ according to Definition 4.1. This allocation principle is the with-without allocation known from Merton and Perold (1993) and Matten (1996). Therefore, if one accepts Definition 4.1 as appropriate for a specific problem-setting the advantage of this allocation, although in general it fails to have the efficiency property, is that it does not require the cost function (accordingly the risk measure with the definition in (4.43)) to have any specific properties. This allocation principle exists whether the underlying risk measure is convex, continuous, Gâteaux-differentiable or not.

The framework of Subsection 3.1 corresponds, in contrast to this Section, to a fractional players framework in game theory (see Aumann and Shapley (1974)) and thus the results obtained in Subsection 3.1 are easily transferable to a game theoretic setting by using a fractional cost and reward function, ie,

$$\begin{aligned} \vartheta(u) &= \theta\left(\sum_{i=1}^{n} u_i X_i\right), \quad u \in U \subset \mathbb{R}^n \\ c(u) &= \rho\left(\sum_{i=1}^{n} u_i X_i\right), \quad u \in U \subset \mathbb{R}^n. \end{aligned}$$

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